Decision Networks and VPI

$$EU(a|e) = \sum_s P(s|e)\,U(a,s)$$

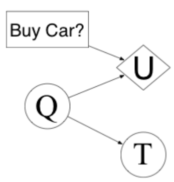
$$MEU(e) = \max_a EU(a|e)$$

$$VPI(e') = MEU(e,e') - MEU(e)$$

CS188 Summer 2018 Section 9: Decision Nets and HMMs

1 Decision Networks and VPI

A used car buyer can decide to carry out various tests with various costs (e.g., kick the tires, take the car to a qualified mechanic) and then, depending on the outcome of the tests, decide which car to buy. We will assume that the buyer is deciding whether to buy car c and that there is time to carry out at most one test which costs \$50 and which can help to figure out the quality of the car. A car can be in good shape (of good quality $Q = +q$) or in bad shape (of bad quality Q=¬q), and the test might help to indicate what shape the car is in. There are only two outcomes for the test T: pass (T=pass) or fail (T=fail). Car c costs \$1,500, and its market value is \$2,000 if it is in good shape; if not, \$700 in repairs will be needed to make it in good shape. The buyers estimate is that c has 70% chance of being in good shape. The Decision Network is shown below.



1. Calculate the expected net gain from buying car c, given no test.

buy car

70% good shape → $U(Q=+q, \text{buy}) = \text{value} - c = 2000 - 1500 = 500$ \$

30% bad shape → $U(Q=\neg q, \text{buy}) = \text{value} - c = 2000 - 2200 = -200$ \$

$$\text{Expected utility} = \mathbb{E}\,U[\text{buy}] = \sum_q P(Q=q)\,U(q,\text{buy})$$

$$= P(Q=+q)\,U(Q=+q,\text{buy}) + P(Q=\neg q)\,U(Q=\neg q,\text{buy})$$

$$= 0.7\,(500) + 0.3\,(-200) = 290$$

2. Tests can be described by the probability that the car will pass or fail the test given that the car is in good or bad shape. We have the following information:

$$P(T = \text{pass}|Q = +q) = 0.9$$
$$P(T = \text{pass}|Q = \neg q) = 0.2$$

Calculate the probability that the car will pass (or fail) its test, and then the probability that it is in good (or bad) shape given each possible test outcome.

$$P(T=\text{pass}) = \sum_q P(T=\text{pass}, Q=q)$$

$$= \sum_q P(T=\text{pass}\,|\,Q=q)\,P(Q=q)$$

$$= P(T=\text{pass}\,|\,Q=+q)\,P(Q=+q) + P(T=\text{pass}\,|\,Q=\neg q)\,P(Q=\neg q)$$

$$= (0.9)(0.7) + (0.2)(0.3)$$

$$= 0.69$$

$$P(T=\text{fail}) = 1 - 0.69 = 0.31$$

$$P(Q=+q\,|\,T=\text{pass}) = \frac{P(T=\text{pass}\,|\,Q=+q)\,P(Q=+q)}{P(T=\text{pass})} = \frac{(0.9)(0.7)}{0.69} = 0.91$$

$$P(Q=+q\,|\,T=\text{fail}) = \frac{P(T=\text{fail}\,|\,Q=+q)\,P(Q=+q)}{P(T=\text{fail})} = \frac{(0.1)(0.7)}{0.31} = 0.22$$

"prob. of ... conditional on being tall/short"

3. Calculate the optimal decisions given either a pass or a fail, and their expected utilities.

How do we know which decision is optimal?

from before

$$\mathbb{E}U(a \mid e) = \sum_{s} P(s \mid e)\, U(a, s)$$

$$\mathbb{E}U(buy \mid T = pass) = \sum_{q} P(Q = q \mid T = pass)\, U(q, buy)$$

$$= P(Q = +q \mid T = pass)\, U(+q, buy) + P(Q = \neg q \mid T = pass)\, U(\neg q, buy)$$

$$= 0.91\,(500) + 0.09\,(-200) \approx 437$$

$$\mathbb{E}U(buy \mid T = fail)$$

$$= P(Q = +q \mid T = fail)\, U(+q, buy) + P(Q = \neg q \mid T = fail)\, U(\neg q, buy)$$

$$= 0.22\,(500) + 0.78\,(-200) = -46$$

$$\mathbb{E}U(\neg buy \mid T = pass) = 0$$

$$\mathbb{E}U(\neg buy \mid T = fail) = 0$$

$$MEU(T = pass) = \max_{a} \mathbb{E}U(a \mid T = pass) = 437 \qquad (buy)$$

$$MEU(T = fail) = \max_{a} \mathbb{E}U(a \mid T = fail) = 0 \qquad (\neg buy)$$

4. Calculate the value of (perfect) information of the test. Should the buyer pay for a test?

from intro

$$\boxed{VPI = MEU(e, e') - MEU(e)}$$

$$VPI = MEU(T) - MEU(\emptyset)$$

$$= \left(\sum_t P(T=t)\, MEU(T=t) \right) - MEU(\emptyset)$$

$$= 0.69\,(437) + 0.31\,(0) - 290 \approx 11.53$$

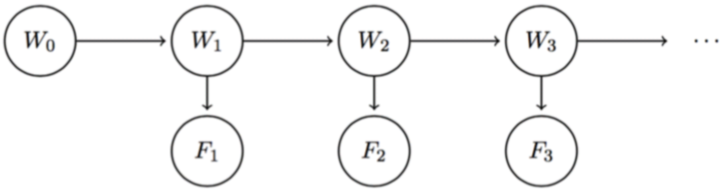
Should buyer get test?

No, as $50\,\$ > 11.53\,\$$

cost   expected gain

HMMs

check out the wikipedia article for forward algorithm! (and note)



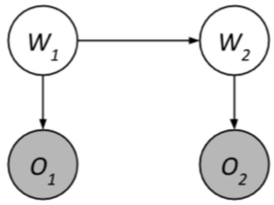
$$
\begin{aligned}
F_1 &\perp\!\!\!\perp W_0 | W_1 \\
\forall i &= 2,\dots,n; \quad W_i \perp\!\!\!\perp \{W_0,\dots,W_{i-2},F_1,\dots,F_{i-1}\} | W_{i-1} \\
\forall i &= 2,\dots,n; \quad F_i \perp\!\!\!\perp \{W_0,\dots,W_{i-1},F_1,\dots,F_{i-1}\} | W_i
\end{aligned}
$$

Goal of forward alg: compute belief

$$B(W_t) = P(W_t | F_{1:t})$$

2 HMMs

Consider the following Hidden Markov Model.



W_1	$P(W_1)$
0	0.3
1	0.7

W_t	W_{t+1}	$P(W_{t+1} \mid W_t)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

W_t	O_t	$P(O_t \mid W_t)$
0	A	0.9
0	B	0.1
1	A	0.5
1	B	0.5

Suppose that we observe $O_1 = A$ and $O_2 = B$.
Using the forward algorithm, compute the probability distribution $P(W_2 \mid O_1 = A, O_2 = B)$ one step at a time.

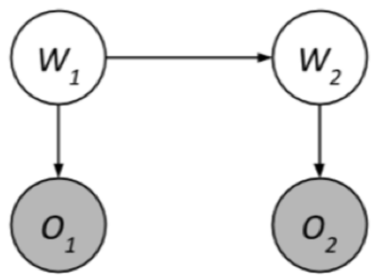
1. Compute $P(W_1, O_1 = A)$.

$$P(W_1, O_1 = A) = P(O_1 = A \mid W_1)\, P(W_1)$$

$$= \begin{cases} (0.9)(0.3) = 0.27 & \text{if } W_1 = 0 \\ (0.5)(0.7) = 0.35 & \text{if } W_1 = 1 \end{cases}$$

2. Using the previous calculation, compute $P(W_2, O_1 = A)$.

$$P(W_2, O_1 = A) = \sum_{w_1} P(W_2, w_1, O_1 = A)$$



$$= \sum_{w_1} P(W_2 \mid w_1, O_1 = A)\, P(w_1, O_1 = A)$$

$$= \sum_{w_1} P(W_2 \mid w_1)\, P(w_1, O_1 = A)$$

$$= P(W_2 \mid W_1 = 0)\, P(W_1 = 0, O_1 = A) + P(W_2 \mid W_1 = 1)\, P(W_1 = 1, O_1 = A)$$

$$= \begin{cases} (0.4)(0.27) + (0.8)(0.35) = 0.388 & \text{if } W_2 = 0 \\ (0.6)(0.27) + (0.2)(0.35) = 0.232 & \text{if } W_2 = 1 \end{cases}$$

3. Using the previous calculation, compute $P(W_2, O_1 = A, O_2 = B)$.

We have:

$W_{t+1} \mid W_t$

$O_t \mid W_t$

W_t

W_t, O_t

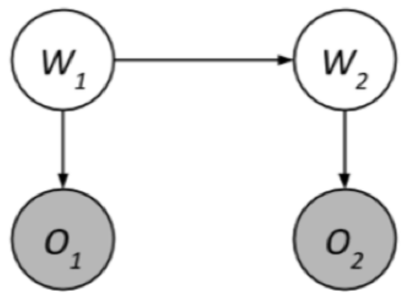
W_{t+1}, O_t

$$P(W_2, O_1 = A, O_2 = B)$$

$$= P(O_2 = B \mid W_2, O_1 = A)\, P(W_2, O_1 = A)$$

$$= P(O_2 = B \mid W_2)\, P(W_2, O_1 = A)$$

$$= \begin{cases} (0.388)(0.1) = 0.0388 & \text{if } W_2 = 0 \\ (0.232)(0.5) = 0.116 & \text{if } W_2 = 1 \end{cases}$$



4. Finally, compute $P(W_2 | O_1 = A, O_2 = B)$.

$$P(W_2 \mid O_1 = A, O_2 = B) = \frac{P(W_2, O_1 = A, O_2 = B)}{P(O_1 = A, O_2 = B)}$$

$$= \frac{P(W_2, O_1 = A, O_2 = B)}{\sum_{w_2} P(w_2, O_1 = A, O_2 = B)}$$

$$= \begin{cases} \dfrac{0.0388}{0.116 + 0.0388} = 0.25 & \text{if } W_2 = 0 \\[2ex] \dfrac{0.116}{0.116 + 0.0388} = 0.75 & \text{if } W_2 = 1 \end{cases}$$

Note this is also

$$= \frac{P(O_2 = B \mid W_2)\, P(W_2, O_1 = A)}{\sum_{w_2} P(w_2, O_1 = A, O_2 = B)}$$ from part 3

$$\propto P(O_2 = B \mid W_2) \sum_{w_1} P(W_2 \mid w_1)\, P(w_1, O_1 = A)$$ from part 2

$$\propto P(O_2 = B \mid W_2) \sum_{w_1} P(W_2 \mid w_1) \underbrace{P(O_1 = A \mid W_1)\, P(W_1)}_{\propto B(W_1)}$$ from part 1

Note that this is in the same form as:

$$\boxed{B(W_{i+1}) \propto Pr(f_{i+1} | W_{i+1}) \sum_{w_i} Pr(W_{i+1} | w_i) B(w_i)}$$

We've basically derived the forward alg. for this specific instance of the graph.

Note that there are parallels between the individual steps of the exercise above and the time elapse update and observation update steps of the forward algorithm